

## NATURAL NUMEROSITIES OF SETS OF TUPLES

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**ABSTRACT.** We consider a notion of “numerosity” for sets of tuples of natural numbers, that satisfies the *five common notions* of Euclid’s Elements, so it can agree with cardinality only for finite sets. By suitably axiomatizing such a notion, we show that, contrasting to cardinal arithmetic, the natural “Cantorian” definitions of order relation and arithmetical operations provide a very good algebraic structure. In fact, numerosities can be taken as the non-negative part of a *discretely ordered ring*, namely the quotient of a formal power series ring modulo a suitable (“gauge”) ideal. In particular, special numerosities, called “natural”, can be identified with the semiring of hypernatural numbers of appropriate ultrapowers of  $\mathbb{N}$ .

### INTRODUCTION

In this paper we consider a notion of “equinumerosity” on sets of tuples of natural numbers, *i.e.* an equivalence relation, finer than equicardinality, that satisfies the celebrated five Euclidean common notions about *magnitudines* ([8]), including the principle that “the whole is greater than the part”. This notion preserves the basic properties of equipotency for *finite* sets. In particular, the natural Cantorian definitions endow the corresponding numerosities with a structure of *discretely ordered semiring*, where 0 is the size of the emptyset, 1 is the size of every singleton, and greater numerosity corresponds to supersets.

This idea of “Euclidean” numerosity has been recently investigated by V. Benci, M. Di Nasso and M. Forti in a sequel of papers, starting with [1] (see also [3]). In particular, in [2] the size of arbitrary sets (of ordinals) was approximated by means of directed unions of finite sets, while in [7] finite dimensional point sets over the real line, and in [4] entire “mathematical universes” are considered.

Here we focus on *sets of tuples of natural numbers*, and we show that the existence of equinumerosity relations for such sets is equivalent to the existence of a class of *prime ideals*, named “gauge”, of a special ring of *formal power series in countably many indeterminates*. Similarly to all papers quoted above, special ultrafilters are used in order to model numerosities. In fact, the gauge ideals corresponding to the equinumerosities called “natural”, are in biunique correspondence with *special ultrafilters* over the set  $\mathbb{N}^{<\omega}$  of all finite subsets of  $\mathbb{N}$ . These ultrafilters, also called “gauge”, may be of independent interest, being not clear their connection with other classes of special ultrafilters considered in the literature. In fact, we only prove here that all selective ultrafilters are gauge.

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The paper is organized as follows. In Section 1 we derive from the Euclidean common notions a few set theoretic principles that are the basis of our formal definition of *equinumerosity relation*. Then we prove that the corresponding *numerocities* form an ordered semiring that can be obtained as the non-negative part of the quotient of a ring  $\mathcal{R}$  of formal power series in countably many indeterminates, modulo suitable prime ideals. In Section 2 we show how numerocities can be embedded in rings of *hyperreal numbers* obtained by means of special ultrafilters. In particular, all “natural” numerocities are essentially *nonstandard natural numbers*. Actually, they arise through isomorphisms with ultrapowers  $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}^{<\omega}}$  modulo “gauge ultrafilters”. A few final remarks and open questions are contained in Section 3.

In general, we refer to [6] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models that are used in this paper.

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## 1. EQUINUMEROSITY OF POINT SETS

In this section we study a notion of “numerosity” for *point sets of natural numbers*, i.e. for sets of tuples of natural numbers. This numerosity will be defined by starting from an equivalence relation of “equinumerosity”, denoted by  $\approx$ , that satisfies the basic properties of *equipotency* between finite sets.

In particular we assume first that equinumerosity satisfies the following<sup>1</sup>

**AP (Aristotelian Principle):**  $A \approx B$  if and only if  $A \setminus B \approx B \setminus A$ .

The axiom AP is a compact equivalent set theoretic formulation of the second and third common notions of Euclid’s Elements: “If equals be added to equals, the wholes are equal”, and “if equals be subtracted from equals, the remainders are equal”. (A precise statement of this equivalence is given in Proposition 1.7 below.) On the other hand, for infinite sets, the Aristotelian Principle is clearly incompatible with the Cantorian notion of equicardinality.

Notice that the first common notion “things which are equal to the same thing are also equal to one another” is already secured by the assumption that equinumerosity is an equivalence relation.

Together with the notion of “having the same numerosity”, it is natural to introduce a “comparison of numerosities”, so as to satisfy the fifth Euclidean common notion “the whole is greater than the part”. Of course, this comparison must be coherent with equinumerosity, so we are led to the following

**Definition** We say that  $A$  is *greater than*  $B$ , denoted by  $A \succ B$ , or equivalently that  $B$  is *smaller than*  $A$ , denoted by  $B \prec A$ , if there exist  $A', B'$  such that  $A' \supset B'$ ,  $A \approx A'$ , and  $B \approx B'$ .

The natural idea that numerosities of sets are always comparable, combined with the fifth Euclidean notion, gives the following trichotomy property:

**ZP (Zermelian Principle):** Exactly one of the following three conditions holds:

- (a)  $A \approx B$ ;
- (b)  $A \succ B$ ;
- (c)  $A \prec B$ .

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<sup>1</sup> We call “Aristotelian” this principle, because the favourite example of “axiom” quoted by Aristotle is “if equals be subtracted from equals, the remainders are equal”.

We shall see below that the Zermelian Principle implies that given two sets one is equinumerous to some superset of the other, and obviously that no proper subset is equinumerous to the set itself. (See Proposition 1.4).

We stress that both properties AP and ZP hold for equipotency between finite sets, while both fail badly for equipotency between infinite sets. So we cannot assume that equipotent sets are always equinumerous, but we have to give a suitable interpretation of the fourth Euclidean common notion “*things applying [exactly] onto one another are equal to one another*”.<sup>2</sup> We choose two kinds of numerosity preserving bijections, namely “permutations of components” and “rising dimension”

**TP (Transformation Principle):** *If  $T$  is 1-1, and  $T(a) = (a_{\tau 1}, \dots, a_{\tau k})$  is a permutation of  $a = (a_1, \dots, a_k)$  for all  $a \in A$ , then  $A \approx T[A]$ .*

**UP (Unit Principle):**  $A \times \{n\} \approx A$  for all  $n \in \mathbb{N}$ .

Remark that the Unit Principle cannot be consistently assumed for all point sets  $A$ . In fact, if  $A = \{n, (n, n), (n, n, n), \dots\}$ , then  $A \times \{n\}$  is a *proper subset* of  $A$ , and so it cannot be equinumerous to  $A$ . So we have to restrict the principle UP. In view of the further developments, in particular in order to obtain a *semiring* of numerosities, a convenient choice is the family of all “finitary” point sets, *i.e.* sets that contain only finitely many tuples for each finite set of components. Denote by

$$\mathbb{W} = \{A \subseteq \mathcal{P}(\bigcup_{k \in \mathbb{N}} \mathbb{N}^k) \mid \forall n \exists h \forall k > h (A \cap \{0, \dots, n\}^k = \emptyset)\}$$

the family of all *finitary point sets*. Remark that  $\mathbb{W}$  is a proper superset of the family of all “finite dimensional” point sets  $\mathbb{W}_0 = \bigcup_{d \in \mathbb{N}} \mathcal{P}(\bigcup_{k=1}^d \mathbb{N}^k)$  that has been considered in [5, 9].

Finally, in order to define a product of numerosities, we could introduce the following principle

**PP (Product Principle):** *If  $A \approx A'$  and  $B \approx B'$  then  $A \times B \approx A' \times B'$ .*

In order to make  $\mathbb{W}$  closed under Cartesian products, we follow the usual practice, and we identify Cartesian products with the corresponding “concatenations”. That is, for every  $A \subseteq \mathbb{N}^k$  and for every  $B \subseteq \mathbb{N}^h$ , we identify

$$A \times B = \{((a_1, \dots, a_k), (b_1, \dots, b_h)) \mid (a_1, \dots, a_k) \in A, (b_1, \dots, b_h) \in B\}$$

with:

$$A \times B = \{(a_1, \dots, a_k, b_1, \dots, b_h) \mid (a_1, \dots, a_k) \in A \text{ and } (b_1, \dots, b_h) \in B\}.$$

With this convention, we have that  $A \times \{x_1, x_2, \dots, x_k\} = A \times \{x_1\} \times \{x_2\} \times \dots \times \{x_k\}$ , and so, using also the Transformation Principle, we obtain the general property

$$\{P\} \times A \approx A \times \{P\} \approx A \text{ for any point } P \in \mathbb{N}^k.$$

In particular, any two singletons are equinumerous.

However, assuming this convention, the Product Principle cannot be consistently postulated in the above formulation for all sets in  $\mathbb{W}$ , because different pairs of tuples may share the same concatenation. *E.g.* both  $((1, 2), (3, 4, 5))$  and  $((1, 2, 3), (4, 5))$  produce  $(1, 2, 3, 4, 5)$ . So one should consider concatenated products as “multisets”, where each tuple comes with its (finite) “multiplicity”. We

<sup>2</sup> See the accurate discussion of this Euclidean common notion by T. Heath in [8].

prefer to consider only pure sets, so we restrict the Product Principle to suitably defined “multipliable pairs”.

Let us call the sets  $A, B \in \mathbb{W}$  *multipliable* if different pairs  $(a, b) \in A \times B$  have different concatenations. (For instance, every set of  $\mathbb{W}_0$  is multipliable with every subset of  $\mathbb{N}^k$ .) We shall restrict PP to products of *multipliable* sets. We can now give our precise definition of equinumerosity relation.

**Definition 1.1.** Let

$$\mathbb{W} = \{A \subseteq \mathcal{P}(\bigcup_{k \in \mathbb{N}} \mathbb{N}^k) \mid \forall n \exists h \forall k > h (A \cap \{0, \dots, n\}^k = \emptyset)\}$$

be the set of all *finitary point sets*.

• An equivalence relation  $\approx$  on  $\mathbb{W}$  is an *equinumerosity* if the following properties are fulfilled for all  $A, B \in \mathbb{W}$ :

AP:  $A \approx B$  if and only if  $A \setminus B \approx B \setminus A$ .

ZP: Exactly one of the following three conditions holds:

$$\text{either } A \approx B, \text{ or } A \succ B, \text{ or } A \prec B,$$

where  $A$  is *greater than*  $B$ , denoted by  $A \succ B$ , or equivalently  $B$  is *smaller than*  $A$ , denoted by  $B \prec A$ , if there exist  $A', B'$  such that

$$A' \supset B', A \approx A', \text{ and } B \approx B'.$$

TP: If  $T$  is 1-1, and  $T(a)$  is a permutation of  $a$  for all  $a \in A$  then  $A \approx T[A]$ .

UP:  $A \times \{n\} \approx A$  for all  $n \in \mathbb{N}$ .

PP: If  $A, B$  and  $A', B'$  are multipliable pairs and  $A \approx A', B \approx B'$  then

$$A \times B \approx A' \times B'.$$

**Definition 1.2.** Let  $\approx$  be an equinumerosity relation on the set  $\mathbb{W}$ .

• The *numerosity* of  $A$  (w.r.t.  $\approx$ ) is the equivalence class  $[A]_{\approx} = \{B \in \mathbb{W} \mid B \approx A\}$  of all point sets equinumerous to  $A$ , denoted by  $\mathfrak{n}_{\approx}(A)$ .

• The *set of numerosities* of  $\approx$  is the quotient set  $\mathfrak{N}_{\approx} = \mathbb{W} / \approx$ , and

• the *numerosity function* associated to  $\approx$  is the canonical map  $\mathfrak{n}_{\approx} : \mathbb{W} \rightarrow \mathfrak{N}_{\approx}$ .

We drop the subscript  $\approx$  whenever the equinumerosity relation is fixed.

Clearly the Unit Principle formalizes the natural idea that singletons have “unitary” numerosity. A trivial but important consequence of this axiom is the existence of infinitely many pairwise disjoint equinumerous copies of any point set. Moreover, infinitely many of them can be taken multipliable with any fixed set of  $\mathbb{W}$ . Namely

**Proposition 1.3.** Let  $A, B \in \mathbb{W}$  be point sets. For  $m, n, h, k \in \mathbb{N}$  put

$$A(m^h, n^k) = A \times \{m\}^h \times \{n\}^k.$$

Assuming UP the sets  $A(m^h, n^k)$  are equinumerous to  $A$  for all  $h, k$ , and are pairwise disjoint, disjoint from  $B$ , and multipliable with  $B$  for all sufficiently large  $h, k$ .

**Proof.** The first assertion is obvious. In order to prove the remaining ones, put  $p = \max\{m, n\}$  and assume that  $A \cap \{0, \dots, p\}^l = \emptyset$  for  $l \geq j$ . Then  $A(m^h, n^k)$  is disjoint from  $A$  and multipliable with  $B$  whenever  $h, k \geq j$ . Moreover  $A(m^h, n^k) \cap A(m^{h'}, n^{k'}) = \emptyset$  whenever  $h, k, h', k' \geq j$  and  $k \neq k'$ .  $\square$

In the following proposition we list several important properties of the binary relation  $\prec$ .

**Proposition 1.4.**

- (i)  $A \prec B$  holds if and only if  $B$  is equinumerous to a proper superset  $B'$  of  $A$ . Hence, given two sets in  $\mathbb{W}$ , one is equinumerous to a superset of the other one.
- (ii) The relation  $\prec$  is a preorder on  $\mathbb{W}$  that induces a total ordering on the quotient set  $\mathfrak{N} = \mathbb{W}/\approx$ .

**Proof.** (i) If there exists a proper superset  $B'$  of  $A$  that is equinumerous to  $B$ , then, from the definition of  $\prec$ , we conclude that  $A \prec B$ . Conversely, suppose that  $A \prec B$ , that is there exist sets  $A'$  and  $B'$  such that  $A' \subset B'$ ,  $A \approx A'$  and  $B \approx B'$ . By possibly applying Proposition 1.3, we may assume without loss of generality that  $A \cap A' = A \cap B' = \emptyset$ . Put  $C = B' \setminus A'$ , and consider  $A \cup C$ , which is a proper superset of  $A$ . By AP we have

$$A \cup C \approx B \iff A \cup C \approx B' \iff A = (A \cup C) \setminus B' \approx B' \setminus (A \cup C) = A'.$$

So  $A \cup C$  is a proper superset of  $A$  equinumerous to  $B$ .

- (ii) Clearly  $A \not\prec A$  because  $A$  cannot be equinumerous to a proper superset of itself. In order to prove transitivity of the relation  $\prec$ , we state the following lemma:

**Lemma 1.5.** *Let  $\approx$  be a relation of equinumerosity on  $\mathbb{W}$ , and let  $A \approx B$ . Then for each proper superset  $A'$  of  $A$  there exists a proper superset  $B'$  of  $B$  such that  $A' \approx B'$ .*

**Proof.** According to (i), let us consider the three possible cases:

- (1)  $A' \approx B$ : then  $A' \approx A$  against ZP.
- (2) There exists a proper superset  $A''$  of  $A'$  such that  $A'' \approx B$ : then  $A$  would be equinumerous to the proper superset  $A''$ , again contradicting ZP.
- (3) There exists a proper superset  $B'$  of  $B$  such that  $A' \approx B'$  and the lemma is proved.  $\square$

Now we can prove transitivity of the relation  $\prec$ . Assume that  $A \prec B$  and  $B \prec C$ : then there exist proper supersets  $A', B'$  of  $A, B$  respectively, such that  $A' \approx B$  and  $B' \approx C$ . Then, by Lemma 1.5, there exists a proper superset  $A''$  of  $A$  such that  $A'' \approx B' \approx C$  and so  $A \prec C$ .  $\square$

Now we prove that the principle AP is equivalent to the conjunction of the second and third common notions of Euclid, when formalized in the following way:

**SP (Sum Principle):** Let  $A, A', B, B' \in \mathbb{W}$  be such that  $A \cap B = \emptyset$  and  $A' \cap B' = \emptyset$ . If  $A \approx A'$  and  $B \approx B'$ , then  $A \cup B \approx A' \cup B'$ .

**DP (Difference Principle):** Let  $A, A', C, C' \in \mathbb{W}$  be such that  $A \subseteq C$  and  $A' \subseteq C'$ . If  $A \approx A'$  and  $C \approx C'$ , then  $C \setminus A \approx C' \setminus A'$ .

This equivalence has been proved in [9], and also in [5] for a slightly different notion of equinumerosity. We repeat the proof here for convenience of the reader, because we need these facts in the sequel. We begin by stating the following lemma:

**Lemma 1.6.** *Let  $\approx$  be an equivalence relation for which AP holds and let  $A, B, A', B' \in \mathbb{W}$  be such that  $B \subseteq A$  and  $B' \subseteq A'$ . If  $B \approx B'$ , then*

$$A \setminus B \approx A' \setminus B' \iff A \approx A'.$$

**Proof.** Put  $B_0 = B \setminus A'$ ,  $B'_0 = B' \setminus A$ ,  $C = B \cap B'$ ,  $B_1 = B \setminus (B_0 \cup C)$ ,  $B'_1 = B' \setminus (B'_0 \cup C)$ ,  $C_0 = A \setminus (B \cup A')$ ,  $C'_0 = A' \setminus (B' \cup A)$ ,  $E = (A \cap A') \setminus (B \cup B')$ . So we obtain pairwise disjoint sets  $B_0, B'_0, C, B_1, B'_1, C_0, C'_0, E$  such that  $B = B_0 \cup B_1 \cup C$ ,

$B' = B'_0 \cup B'_1 \cup C$ ,  $A \setminus A' = B_0 \cup C_0$ ,  $A' \setminus A = B'_0 \cup C'_0$ ,  $A \setminus B = B'_1 \cup C_0 \cup E$ ,  
 $A' \setminus B' = B_1 \cup C'_0 \cup E$ . By AP we can write

$$A \approx A' \iff B_0 \cup C_0 = A \setminus A' \approx A' \setminus A = B'_0 \cup C'_0,$$

$$A \setminus B \approx A' \setminus B' \iff B'_1 \cup C_0 = (A \setminus B) \setminus (A' \setminus B') \approx (A' \setminus B') \setminus (A \setminus B) = B_1 \cup C'_0.$$

By hypothesis and AP we can write

$$B_0 \cup B_1 \cup C = B \approx B' = B'_0 \cup B'_1 \cup C \implies B_0 \cup B_1 \approx B'_0 \cup B'_1$$

and hence

$$B_0 \cup B_1 \cup C_0 \approx B'_0 \cup B'_1 \cup C_0.$$

Suppose that  $A \setminus B \approx A' \setminus B'$ , that is  $B'_1 \cup C_0 \approx B_1 \cup C'_0$ , it follows that  $B'_0 \cup B'_1 \cup C_0 \approx B'_0 \cup B_1 \cup C'_0$ , hence  $B_0 \cup B_1 \cup C_0 \approx B'_0 \cup B_1 \cup C'_0$  and we conclude  $A \approx A'$ .

Conversely, if  $A \approx A'$ , that is  $B_0 \cup C_0 \approx B'_0 \cup C'_0$ , we have  $B_0 \cup B_1 \cup C_0 \approx B'_0 \cup B_1 \cup C'_0$ , hence  $B'_0 \cup B'_1 \cup C_0 \approx B'_0 \cup B_1 \cup C'_0$  and we conclude  $A \setminus A' \approx A' \setminus A$ .  $\square$

**Proposition 1.7.** *Let  $\approx$  be an equivalence relation. The Axiom AP is equivalent to the conjunction of the two principles SP and DP.*

**Proof.** The conjunction of SP and DP yields AP, namely

$$(A \setminus A') \cup (A \cap A') = A \approx A' = (A' \setminus A) \cup (A \cap A') \implies^{DP} (A \setminus A') \approx (A' \setminus A),$$

$$A \setminus (A \cap A') = (A \setminus A') \approx (A' \setminus A) = A' \setminus (A \cap A') \implies^{SP} A \approx A'.$$

Conversely, assume AP: then, by Lemma 1.6, both SP and DP hold.  $\square$

We prove now that our notion of equinumerosity satisfies what can be viewed as a necessary condition, *videlicet that finite point-sets are equinumerous if and only if they have the same “number of elements”*.

**Proposition 1.8.** *Let  $\approx$  be an equinumerosity relation, and let  $A, B \in \mathbb{W}$  be finite sets. Then*

$$A \approx B \iff |A| = |B|.$$

*Moreover, if  $X$  is infinite, then  $X \succ A$ . Hence  $\mathbb{N}$  can be taken as an initial segment of the set of numerosities  $\mathfrak{N}$  corresponding to  $\approx$ .*

**Proof.** First observe that  $\emptyset$ , being a proper subset of any nonempty set  $A$ , cannot be equinumerous to  $A$ .

Secondly, we have already remarked that any two singletons  $\{a\}, \{b\}$  are equinumerous. Moreover, if  $C \approx \{b\}$ , then  $C$  is a singleton. In fact let  $c$  be an element of  $C$ ; then  $\{c\} \approx \{b\} \approx C$ , hence  $\{c\} = C$ , because  $C$  cannot be a proper superset of  $\{c\}$ .

Finally, given two finite sets  $A$  and  $B$ , we proceed by induction on  $k$ , the least cardinality of the sets  $A, B$ . The case  $k = 1$  has already been dealt with. Assume the thesis true for  $k \leq n$  and let  $A$  and  $B$  be finite sets such that  $n+1 = \min \{|A|, |B|\}$ . Pick  $a \in A$  and  $b \in B$ , and put  $A' = A \setminus \{a\}$ ,  $B' = B \setminus \{b\}$ . Since  $\{a\} \approx \{b\}$ , Lemma 1.6 and the induction hypothesis yield

$$A \approx B \iff A' \approx B' \iff |A'| = |B'| \iff |A| = |B|.$$

Now if  $X$  is an infinite set and  $A$  is a finite set, we can find a proper subset  $B$  of  $X$ , such that  $|A| = |B|$ , so we conclude that  $A \prec X$ .  $\square$

By the above proposition, we can identify each natural number  $n \in \mathbb{N}$ , with the equivalence class of all those point sets that have finite cardinality  $n$ .

Starting from the equivalence relation of *equipotency*, Cantor introduced the algebra of cardinals by means of disjoint unions and Cartesian products. So we could similarly introduce an algebra on “numerosities”. The given axioms have been chosen so as to guarantee that numerosities are naturally equipped with a “nice” algebraic structure. (This is to be contrasted with the awkward cardinal algebra, where *e.g.*  $\kappa + \mu = \kappa \cdot \mu = \max \{\kappa, \mu\}$  for all infinite  $\kappa, \mu$ .)

**Theorem 1.9.** *Let  $\mathfrak{N}$  be the set of numerosities of the equinumerosity relation  $\approx$ . Then there exist unique operations  $+$  and  $\cdot$ , and a unique linear order  $<$  on  $\mathfrak{N}$ , such that for all point sets  $X, Y \in \mathbb{W}$ :*

- (1)  $\mathfrak{n}(X) + \mathfrak{n}(Y) = \mathfrak{n}(X \cup Y)$  whenever  $X \cap Y = \emptyset$ ;
- (2)  $\mathfrak{n}(X) \cdot \mathfrak{n}(Y) = \mathfrak{n}(X \times Y)$  whenever  $X, Y$  are multipliable;
- (3)  $\mathfrak{n}(X) < \mathfrak{n}(Y)$  if and only if  $Y \approx Y'$  for some proper superset  $Y' \supset X$ .

*The resulting structure on  $\mathfrak{N}$  is the non-negative part of a discretely ordered ring  $(\mathfrak{N}, 0, 1, +, \cdot, <)$ . Moreover, if the fundamental subring of  $\mathfrak{N}$  is identified with  $\mathbb{Z}$ , then  $\mathfrak{n}(X) = |X|$  for every finite point set  $X$ .*

We could prove the above theorem by the very same arguments used in [5] or in [9]. However we prefer to obtain a more precise algebraic characterization of the arithmetic of numerosities. To this aim, we consider a suitable ring of formal power series with integer coefficients, and we prove that the set of numerosities can be identified with the non-negative part of the quotient of this ring modulo a suitable prime ideal.

- Let  $\mathbb{T} = \langle t_n \mid n \in \mathbb{N} \rangle$  be a sequence of indeterminates. Let  $\mathbf{A}$  be the set of all eventually zero sequences  $\mathbf{a} = (a_0, a_1, \dots)$  of non-negative integers, and denote by  $t^{\mathbf{a}}$  the monomial  $\prod_{i \in \mathbb{N}} t_i^{a_i}$ .

So any series  $S$  in the variables of  $\mathbb{T}$  can be written as  $S = \sum_{\mathbf{a} \in \mathbf{A}} n_{\mathbf{a}} t^{\mathbf{a}}$  where  $n_{\mathbf{a}}$  is the coefficient of the monomial  $t^{\mathbf{a}}$ .

- Given a point  $x = (x_1, \dots, x_d) \in \mathbb{N}^d$ , consider the sequence  $\mathbf{a} \in \mathbf{A}$ , where  $a_i = |\{j \mid x_j = i\}|$  and associate to  $x$  the monomial  $t_x = t^{\mathbf{a}}$ .
- The *characteristic series* of the nonempty point set  $X \in \mathbb{W}$  is the formal series

$$S_X = \sum_{x \in X} t_x = \sum_{\mathbf{a} \in \mathbf{A}} n_{\mathbf{a}} t^{\mathbf{a}}, \text{ where } n_{\mathbf{a}} = |\{x \in X \mid t_x = t^{\mathbf{a}}\}|.$$

(If  $X = \emptyset$ , put  $S_X = 0$ .)

- Characteristic series behave well with respect to *unions*, *differences* and *products*:

$$S_X + S_Y = S_{X \cup Y} + S_{X \cap Y}, \quad S_X - S_Y = S_{X \setminus Y} \text{ if } Y \subset X$$

$$S_X \cdot S_Y = S_{X \times Y} \text{ if } X, Y \text{ are multipliable.}^3$$

*Remark 1.10.* A series  $S = \sum n_{\mathbf{a}} t^{\mathbf{a}}$  with *non-negative* integer coefficients is the characteristic series of a set  $X \in \mathbb{W}$  if and only if for all  $\mathbf{a} = (a_0, a_1, \dots)$  we have  $n_{\mathbf{a}} \leq \frac{k!}{\prod a_i!}$ , where  $k = \sum_i a_i$ .

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<sup>3</sup> Remark that, if the product  $X \times Y$  is considered as a multiset where each tuple comes with its multiplicity, then the equality holds for all  $X, Y \in \mathbb{W}$ .

In fact, the number of different sequences  $\mathbf{a} = (a_0, a_1, \dots)$  that correspond to the same monomial  $t^{\mathbf{a}}$  is  $\frac{k!}{\prod a_i!}$ , where  $k = \sum_i a_i$  is the degree of the monomial.

- Let  $\mathcal{R}$  be the ring of all formal series of *bounded degree  $d_n$  in each variable  $t_n \in \mathbb{T}$*  with coefficients  $n_{\mathbf{a}}$  such that, for some  $b \in \mathbb{N}$ ,

$$|n_{\mathbf{a}}| \leq b \frac{(\sum_i a_i)!}{\prod a_i!}.$$

- Let  $\mathcal{R}^+$  be the multiplicative subset of the *positive series*, i.e. the series in  $\mathcal{R}$  having only positive coefficients.
- Let  $\mathfrak{I}_0$  be the ideal of  $\mathcal{R}$  generated by  $\{t_n - 1 \mid n \in \mathbb{N}\}$ .

It is easily seen that  $\mathcal{R}$  is the subring with identity of  $\mathbb{Z}[[\mathbb{T}]]$  generated by the set of the characteristic series of all point sets. Moreover every positive series  $P \in \mathcal{R}^+$  is equivalent modulo  $\mathfrak{I}_0$  to some characteristic series.

**Lemma 1.11.** *Every positive series  $P \in \mathcal{R}^+$  is equivalent modulo  $\mathfrak{I}_0$  to the characteristic series of some set in  $\mathbb{W}$ . So any series  $S \in \mathcal{R}$  can be written as  $S = S_X - S_Y + S_0$  for suitable  $X, Y \in \mathbb{W}$  and  $S_0 \in \mathfrak{I}_0$ .*

**Proof.** Suppose that the series  $S$  has non-negative coefficients  $n_{\mathbf{a}} \leq b \frac{k!}{\prod a_i!}$ , where  $k = \sum_i a_i$ . We can decompose  $S$  in at most  $b$  series with coefficients satisfying the conditions for being characteristic, plus a non-negative integer  $a$ . So we can write  $S = a + S_{X_1} + \dots + S_{X_s}$ , where  $s \leq b$  and for all  $i$ ,  $X_i \in \mathbb{W}$ . (The sets  $X_i$  may be not distinct, in general.) According to Proposition 1.3, we can multiply the integer  $a$  and each series  $S_{X_i}$  by suitable monomials  $t_m^h t_n^k$ , so that  $at_m^h t_n^k$  is the characteristic series of a set  $Y_0$  of tuples containing  $h$  times  $m$  and  $k$  times  $n$ , and the remaining series are the characteristic series of pairwise disjoint sets  $Y_i$  equinumerous to  $X_i$ . Clearly, each  $S_{X_i}$  is equivalent modulo  $\mathfrak{I}_0$  to  $S_{Y_i}$ , so putting  $Y = Y_0 \cup Y_1 \cup \dots \cup Y_s$ , we have that  $S$  is equivalent modulo  $\mathfrak{I}_0$  to the characteristic series  $S_Y = S_{Y_0} + \dots + S_{Y_s}$ .

The final assertion of the theorem follows by considering separately the positive and the negative parts of any given series of  $\mathcal{R}$ .  $\square$

In order to classify all equinumerosities, we introduce the following definition:

**Definition 1.12.** Call an ideal  $\mathfrak{I}$  of  $\mathcal{R}$  a *gauge ideal* if

- $\mathfrak{I}_0 \subseteq \mathfrak{I}$ ,
- $\mathcal{R}^+ \cap \mathfrak{I} = \emptyset$ , and
- for all  $S \in \mathcal{R} \setminus \mathfrak{I}$  there exists  $P \in \mathcal{R}^+$  such that either  $S + P \in \mathfrak{I}$  or  $S - P \in \mathfrak{I}$ .

Remark that when  $\mathfrak{I}$  is a gauge ideal the quotient  $\mathcal{R}/\mathfrak{I}$  is a *discretely ordered ring* whose *positive elements* are the cosets  $P + \mathfrak{I}$  for  $P \in \mathcal{R}^+$ . In particular  $\mathfrak{I}$  is a *prime ideal* of  $\mathcal{R}$  that is *maximal* among the ideals disjoint from  $\mathcal{R}^+$ .

Then we have

**Theorem 1.13.** *There exists a biunique correspondence between equinumerosity relations on the space  $\mathbb{W}$  of all point sets over  $\mathbb{N}$  and gauge ideals on the ring  $\mathcal{R}$  of all bounded power series in countably many indeterminates. In this correspondence, if the equinumerosity  $\approx$  corresponds to the ideal  $\mathfrak{I}$ , then*

$$(**) \quad X \approx Y \iff S_X - S_Y \in \mathfrak{I}.$$



More precisely, let  $\mathbf{n} : \mathbb{W} \rightarrow \mathfrak{N}$  be the numerosity function associated to  $\approx$ , and let  $\pi : \mathcal{R} \rightarrow \mathcal{R}/\mathfrak{I}$  be the canonical projection. Then there exists a unique order preserving embedding  $j$  of  $\mathfrak{N}$  onto the non-negative part of  $\mathcal{R}/\mathfrak{I}$  such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{W} & \xrightarrow{\Sigma} & \mathcal{R} \\ \mathbf{n} \downarrow & (*) & \downarrow \pi \\ \mathfrak{N} & \xrightarrow{j} & \mathcal{R}/\mathfrak{I} \end{array}$$

(where  $\Sigma$  maps any  $X \in \mathbb{W}$  to its characteristic series  $S_X \in \mathcal{R}^+$ .)

The ordered semiring structure induced on  $\mathfrak{N}$  by  $j$  satisfies the conditions

- (1)  $\mathbf{n}(X) + \mathbf{n}(Y) = \mathbf{n}(X \cup Y)$  whenever  $X \cap Y = \emptyset$ ;
- (2)  $\mathbf{n}(X) \cdot \mathbf{n}(Y) = \mathbf{n}(X \times Y)$  whenever  $X, Y$  are multipliable;
- (3)  $\mathbf{n}(X) < \mathbf{n}(Y)$  if and only if  $Y \approx Y'$  for some proper superset  $Y' \supset X$ .

**Proof.**

Given a gauge ideal  $\mathfrak{I}$  over  $\mathcal{R}$ , we define the equivalence relation over  $\mathbb{W}$  by condition (\*\*):  $X \approx Y \iff S_X - S_Y \in \mathfrak{I}$ . We prove that the relation  $\approx$  so defined is in fact an equinumerosity relation.

**AP:** We can write  $S_X - S_Y = S_{X \setminus Y} + S_{X \cap Y} - S_{Y \setminus X} - S_{X \cap Y} = S_{X \setminus Y} - S_{Y \setminus X}$ , hence  $X \approx Y \iff X \setminus Y \approx Y \setminus X$ .

**ZP:** Assume first that  $S_X - S_Y \in \mathfrak{I}$ , that is  $X \approx Y$ . If there exists a superset  $Y'$  of  $X$  such that  $S_{Y'} - S_Y \in \mathfrak{I}$ , then  $S_{Y'} - S_X = S_{Y' \setminus X} \in \mathfrak{I} \cap \mathcal{R}^+$ , contradiction. Similarly there exists no superset  $X'$  of  $Y$  equinumerous to  $X$ .

Now assume that  $S_X - S_Y \notin \mathfrak{I}$ , that is  $X \not\approx Y$ ; then there exists a series  $P \in \mathcal{R}^+$  such that either  $S_X - S_Y + P$  or  $S_X - S_Y - P$  belongs to  $\mathfrak{I}$ , by the third property of gauge ideals. Assume without loss of generality that  $S_X - S_Y + P \in \mathfrak{I}$ . Then  $P$  is equivalent modulo  $\mathfrak{I}$  to the characteristic series of a set  $Z$  that we can choose disjoint from  $X$ , by Proposition 1.11. Hence  $S_{X \cup Z} - S_Y = S_X - S_Y + S_Z \in \mathfrak{I}$ , whence  $(X \cup Z) \approx Y$ .

**TP:**  $S_X$  is obviously equal to  $S_{T[X]}$  whenever  $T$  is a transformation that permutes the components of the tuples in  $X$ . Hence  $X \approx T[X]$ .

**UP:** For any set  $X \in \mathbb{W}$ , we have  $S_{X \times \{n\}} - S_X = S_X \cdot (t_n - 1) \in \mathfrak{I}_0 \subseteq \mathfrak{I}$ , i.e.  $X \times \{n\} \approx X$ .

**PP:** Let  $X, Y$  and  $X', Y'$  be multipliable point sets. If  $S_X - S_{X'}$  and  $S_Y - S_{Y'}$  belong to  $\mathfrak{I}$ , that is  $X \approx X'$  and  $Y \approx Y'$ , then

$$S_{X \times Y} - S_{X' \times Y'} = (S_X - S_{X'}) \cdot S_Y + S_{X'} \cdot (S_Y - S_{Y'}) \in \mathfrak{I},$$

hence  $X \times Y \approx X' \times Y'$ .

Conversely, given an equinumerosity relation  $\approx$  over  $\mathbb{W}$ , let  $\mathfrak{I}$  be the ideal of  $\mathcal{R}$  generated by the set  $\{S_X - S_Y \mid X \approx Y\} \cup \{t_0 - 1\}$ . We prove that  $\mathfrak{I}$  is a gauge ideal.

First observe that the ideal  $\mathfrak{I}_0$  is also generated by  $\{t_n - t_0 \mid n \in \mathbb{N}\} \cup \{t_0 - 1\}$ , so  $\mathfrak{I}_0$  is contained in  $\mathfrak{I}$ , because  $t_n - t_0 = S_{\{n\}} - S_{\{0\}}$ .

By Proposition 1.11, given a series  $S \in \mathcal{R}$ , there exist two sets  $X$  and  $Y$  of  $\mathbb{W}$ , such that  $S$  is equivalent modulo  $\mathfrak{I}$  to  $S_X - S_Y$ . If  $S \notin \mathfrak{I}$ , then  $X \not\approx Y$ , so, without loss of generality, we may assume that there exists a superset  $X'$  of  $Y$  such that  $X \approx X'$ , that is  $S_X - S_{X'} \in \mathfrak{I}$ , by ZP. Then we can write  $S_X - S_{X'} = S_X - S_{X' \setminus Y} - S_Y \in \mathfrak{I}$  and hence  $S_X - S_Y$  is congruent modulo  $\mathfrak{I}$  to  $S_{X' \setminus Y} \in \mathcal{R}^+$ .

It remains to prove that  $\mathcal{R}^+ \cap \mathfrak{I} = \emptyset$ . We need the following fact:

**Claim.** *For any  $S \in \mathfrak{I}$  there exist  $h, k \in \mathbb{N}$  and finitely many (not necessarily distinct) sets  $Z_1, W_1, \dots, Z_m, W_m$  of  $\mathbb{W}$ , such that  $Z_1 \approx W_1, \dots, Z_m \approx W_m$  and*

$$t_0^h t_1^k S = (S_{Z_1} - S_{W_1}) + \dots + (S_{Z_m} - S_{W_m}).$$

**Proof.** By definition of  $\mathfrak{I}$ , there exist  $S_0, \dots, S_n \in \mathcal{R}$  such that

$$S = S_0 \cdot (t_0 - 1) + S_1 \cdot (S_{X_1} - S_{Y_1}) + \dots + S_n \cdot (S_{X_n} - S_{Y_n}),$$

with  $X_1 \approx Y_1, \dots, X_n \approx Y_n$ .

According to Proposition 1.11, there exist  $a_i \in \mathbb{Z}$  and  $X_{1i}, \dots, X_{si}; Y_{1i}, \dots, Y_{ti} \in \mathbb{W}$  such that

$$S_i = a_i + S_{X_{1i}} + \dots + S_{X_{si}} - S_{Y_{1i}} - \dots - S_{Y_{ti}}.$$

Then we have

$$\begin{aligned} S_i \cdot (S_{X_i} - S_{Y_i}) &= a_i \cdot (S_{X_i} - S_{Y_i}) + S_{X_{1i}} \cdot (S_{X_i} - S_{Y_i}) + \dots + S_{X_{si}} \cdot (S_{X_i} - S_{Y_i}) \\ &\quad - S_{Y_{1i}} \cdot (S_{X_i} - S_{Y_i}) - \dots - S_{Y_{ti}} \cdot (S_{X_i} - S_{Y_i}). \end{aligned}$$

By Proposition 1.3 we can find  $h, k \in \mathbb{N}$  such that all sets

$$U_{ji} = X_{ji} \times \{0\}^h \times \{1\}^k, V_{li} = Y_{li} \times \{0\}^h \times \{1\}^k$$

are multipliable with both  $X_i$  and  $Y_i$ .

For all  $i > 0$  we have

$$\begin{aligned} t_0^h t_1^k S_i (S_{X_i} - S_{Y_i}) &= a_i t_0^h t_1^k (S_{X_i} - S_{Y_i}) + S_{U_{1i}} (S_{X_i} - S_{Y_i}) + \dots + S_{U_{si}} (S_{X_i} - S_{Y_i}) \\ &\quad - S_{V_{1i}} (S_{X_i} - S_{Y_i}) - \dots - S_{V_{ti}} (S_{X_i} - S_{Y_i}) \\ &= a_i (S_{\{0\}^h \times \{1\}^k \times X_i} - S_{\{0\}^h \times \{1\}^k \times Y_i}) + (S_{U_{1i} \times X_i} - S_{U_{1i} \times Y_i}) \\ &\quad + \dots + (S_{U_{si} \times X_i} - S_{U_{si} \times Y_i}) - (S_{V_{1i} \times X_i} - S_{V_{1i} \times Y_i}) - \dots \\ &\quad - (S_{V_{ti} \times X_i} - S_{V_{ti} \times Y_i}). \end{aligned}$$

Similarly, observing that  $t_0 - 1 = S_{\{0\}} - 1$  we obtain for  $i = 0$

$$\begin{aligned} t_0^h t_1^k S_0 (t_0 - 1) &= a_0 (S_{\{0\}^{h+1} \times \{1\}^k} - S_{\{0\}^h \times \{1\}^k}) + (S_{U_{10} \times \{0\}} - S_{U_{10}}) + \dots \\ &\quad + (S_{U_{s0} \times \{0\}} - S_{U_{s0}}) - (S_{V_{10} \times \{0\}} - S_{V_{10}}) - \dots - (S_{V_{t0} \times \{0\}} - S_{V_{t0}}). \end{aligned}$$

(If  $a_0 > 0$  we take  $a_0$  times the difference  $S_{\{0\}^{h+1} \times \{1\}^k} - S_{\{0\}^h \times \{1\}^k}$ , whereas if  $a_0 < 0$  we take  $-a_0$  times the difference  $S_{\{0\}^h \times \{1\}^k} - S_{\{0\}^{h+1} \times \{1\}^k}$ .)

By PP we have  $U_{ji} \times X_i \approx U_{ji} \times Y_i$  and  $V_{ji} \times X_i \approx V_{ji} \times Y_i$  for all  $j$ , and obviously  $\{0\}^h \times \{1\}^k \times X_i \approx \{0\}^h \times \{1\}^k \times Y_i$ . Hence, by taking  $Z_1, \dots, Z_m$  and  $W_1, \dots, W_m$  to be an enumeration of the sets  $U_{ji} \times X_i, V_{ji} \times X_i, \{0\}^{h+1} \times \{1\}^k$  and  $U_{ji} \times Y_i, V_{ji} \times Y_i, \{0\}^h \times \{1\}^k$ , respectively, we obtain the claim.  $\square$

Clearly,  $t_0^h t_1^k S \in \mathcal{R}^+$  if and only if  $S \in \mathcal{R}^+$ . Towards a contradiction, assume that  $S$  be in  $\mathfrak{I} \cap \mathcal{R}^+$ . By the Claim, there exist  $h, k \in \mathbb{N}$  such that  $t_0^h t_1^k S = (S_{Z_1} - S_{W_1}) + \dots + (S_{Z_m} - S_{W_m})$ , for suitable  $Z_i \approx W_i$ .

Since  $S$  belongs to  $\mathcal{R}^+$ , each  $W_i$  can be decomposed into pairwise disjoint subsets:  $W_i = \bigcup_{j=1}^m W_{ij}$ , in such a way that  $W_{ij} \subseteq Z_j$ . Put  $P = (1, \dots, m)$  and let

$\sigma_1, \dots, \sigma_m$  be different permutations of  $\{1, \dots, m\}$ . Put  $X_i = \{\sigma_i(P)\} \times Z_i$  and  $Y_i = \{\sigma_i(P)\} \times W_i$ . Then we have  $t_P \cdot S_{Z_i} = S_{X_i}$  and  $t_P \cdot S_{W_i} = S_{Y_i}$ , where  $X_1, \dots, X_m$ , and  $Y_1, \dots, Y_m$ , are pairwise disjoint sets of  $\mathbb{W}$ .

Put  $X = \bigcup_{i=1}^m X_i$  and  $Y = \bigcup_{i=1}^m Y_i$ . Then the series

$$t_P t_0^h t_1^k S = (S_{X_1} - S_{Y_1}) + \dots + (S_{X_m} - S_{Y_m}) = (S_X - S_Y)$$

is still in  $\mathcal{R}^+ \cap \mathcal{J}$ . By UP we have  $Z_i \approx X_i \approx Y_i \approx W_i$ , hence  $X \approx Y$ , by SP.

For each monomial  $t^b$  of the series  $S_Y$  there exists a point  $Q \in Y_i$  such that  $t^b = t_Q$ . Note that  $t_Q = t_P t_{Q_i}$ , where  $Q_i$  is a point in  $W_i$ , so, by hypothesis,  $t_{Q_i}$  is also a monomial of some series  $S_{Z_j}$ . Hence there exists a permutation of the coordinates of  $Q$  such that the corresponding point  $Q'$  belongs to  $X_j \subset X$ . Repeating this argument for each monomial of the series  $S_Y$ , we can conclude that there exists a transformation  $T$  of  $Y$ , such that  $T[Y] \subseteq X$  and  $T[Y] \approx Y \approx X$ , hence  $X = T[Y]$ . Then we have  $S_X - S_Y = S_X - S_{T[Y]} = 0$  and so  $(S_{X_1} - S_{Y_1}) + \dots + (S_{X_m} - S_{Y_m}) = 0$ , and this implies  $S = 0$ , absurd. So the proof that  $\mathcal{R}^+ \cap \mathcal{J} = \emptyset$  is complete.

Given an equinumerosity relation  $\approx$ , let  $\varphi(\approx)$  be the gauge ideal generated by the set  $\{S_X - S_Y \mid X \approx Y\} \cup \{t_0 - 1\}$ . The map

$$\varphi: \{\approx \mid \approx \text{equinumerosity relation of } \mathbb{W}\} \rightarrow \{\mathcal{J} \mid \mathcal{J} \text{ gauge ideal of } \mathcal{R}\}$$

is injective: in fact let  $\approx_1$  and  $\approx_2$  be two different equinumerosity relations, then there exist two sets  $X, Y$  of  $\mathbb{W}$ , such that  $X \approx_1 Y$  and  $X \not\approx_2 Y$ . Assume without loss of generality that there exists a proper superset  $Y'$  of  $X$  such that  $Y' \approx_2 Y$ . Put  $\varphi(\approx_1) = \mathcal{J}_1$  and  $\varphi(\approx_2) = \mathcal{J}_2$ : then we have  $S_X - S_Y \in \mathcal{J}_1$  and  $S_{Y'} - S_Y \in \mathcal{J}_2$ . If  $S_X - S_Y$  belongs to  $\mathcal{J}_2$ , then the series  $S_{Y'} - S_X = S_{Y' \setminus X}$  is an element of  $\mathcal{R}^+ \cap \mathcal{J}_2$ , absurd; so  $\mathcal{J}_1 \neq \mathcal{J}_2$  and  $\varphi$  is 1-to-1.

Given a gauge ideal  $\mathcal{J}$ , consider the equinumerosity relation  $\approx$  defined by  $\mathcal{J}$  through (\*\*). Putting  $\varphi(\approx) = \mathcal{J}_\approx$ , we have obviously  $\mathcal{J}_\approx \subseteq \mathcal{J}$ , but, if there exists  $S \in \mathcal{J} \setminus \mathcal{J}_\approx$ , then there exists  $P \in \mathcal{R}^+$  such that either  $S + P$  or  $S - P$  belongs to  $\mathcal{J}_\approx$ . Without loss of generality suppose that  $S + P \in \mathcal{J}_\approx$ : then  $S + P \in \mathcal{J}$  and so  $P \in \mathcal{J}$ , absurd. Therefore  $\mathcal{J}_\approx = \mathcal{J}$  and  $\varphi$  is biunique.

Fixed an equinumerosity relation  $\approx$  and the corresponding gauge ideal  $\mathcal{J}$ , define  $j: \mathfrak{N} \rightarrow \mathcal{R}/\mathcal{J}$  by  $j(\mathbf{n}(X)) = S_X + \mathcal{J}$ . The application  $j$  is well-defined, because, whenever  $X, Y$  are two equinumerous sets of  $\mathbb{W}$ , we have  $S_X - S_Y \in \mathcal{J}$ , hence  $j(\mathbf{n}(X)) = j(\mathbf{n}(Y))$ . Moreover, if  $j(\mathbf{n}(X)) = j(\mathbf{n}(Y))$ , that is  $S_X - S_Y \in \mathcal{J}$ , then  $\mathbf{n}(X) = \mathbf{n}(Y)$ , hence  $j$  is an embedding of  $\mathfrak{N}$  into the non-negative part of  $\mathcal{R}/\mathcal{J}$ . The range of  $j$  is exactly  $(\mathcal{R}^+ + \mathcal{J})/\mathcal{J}$  because each element of  $\mathcal{R}^+$  is congruent modulo  $\mathcal{J}$  to a characteristic series.

We prove that  $j$  induces an ordered semiring structure on  $\mathfrak{N}$ . Given two disjoint sets  $X, Y$  of  $\mathbb{W}$ , define  $\mathbf{n}(X) + \mathbf{n}(Y) = j^{-1}(j(\mathbf{n}(X)) + j(\mathbf{n}(Y)))$ : then

$$\mathbf{n}(X) + \mathbf{n}(Y) = j^{-1}(S_X + S_Y + \mathcal{J}) = j^{-1}(S_{X \cup Y} + \mathcal{J}) = \mathbf{n}(X \cup Y).$$

Similarly, given two multipliable sets  $X, Y$  of  $\mathbb{W}$ , define  $\mathbf{n}(X) \cdot \mathbf{n}(Y) = j^{-1}(j(\mathbf{n}(X)) \cdot j(\mathbf{n}(Y)))$ : then

$$\mathbf{n}(X) \cdot \mathbf{n}(Y) = j^{-1}(S_X \cdot S_Y + \mathcal{J}) = j^{-1}(S_{X \times Y} + \mathcal{J}) = \mathbf{n}(X \times Y).$$

Sum and product are well defined by Proposition 1.3.

If  $X, Y \in \mathbb{W}$  are not equinumerous, *i.e.*  $S_X - S_Y \in \mathcal{R} \setminus \mathcal{J}$ , then there exists a set  $Z \in \mathbb{W}$  such that either  $S_X - S_Y + S_Z$  or  $S_X - S_Y - S_Z$  belongs to  $\mathcal{J}$ , by the third

propriety of gauge ideals. So define a total order  $<$  on  $\mathfrak{N}$  by putting

$$\mathfrak{n}(X) < \mathfrak{n}(Y) \iff \exists Z \in \mathbb{W}, Z \neq \emptyset \text{ such that } S_X - S_Y + S_Z \in \mathfrak{I}.$$

Note that, by Proposition 1.11, if  $\mathfrak{n}(X) < \mathfrak{n}(Y)$  we can always choose the set  $Z$  so that  $X$  and  $Z$  are disjoint and hence  $S_X - S_Y + S_Z = S_{X \cup Z} - S_Y \in \mathfrak{I}$ . Therefore  $\mathfrak{n}(X) < \mathfrak{n}(Y)$  holds if and only if there exists a proper superset  $Y'$  of  $X$  such that  $Y' \approx Y$ . By Proposition 1.4 we conclude that the relation  $<$  is a total order on  $\mathfrak{N}$  and hence  $(\mathfrak{N}, +, \cdot, <)$  is an ordered semiring.  $\square$

*Remark 1.14.* In [9] a class of particular equinumerosity relations has been considered, namely those which are preserved under “natural transformations”, *i.e.* bijections that preserve the *support* (set of components) of each tuple.

Let us call an equinumerosity relation *natural* if it satisfies the following Natural Transformation Principle, which is a severe strengthening of TP:

(NP) If  $T$  is 1-to-1 on  $X \in \mathbb{W}$  and  $\text{supp}(T(x)) = \text{supp}(x)$  for all  $x \in X$ , then  $X \approx T[X]$ .

For  $S \in \mathcal{R}$  let  $S'$  be the *associated squarefree series*, *i.e.* the series obtained by replacing each monomial in  $S$  by the corresponding squarefree monomial and summing up the corresponding coefficients. *I.e.*

$$\text{if } S = \sum_{\mathbf{a} \in \mathbf{A}} n_{\mathbf{a}} t^{\mathbf{a}} \text{ then } S' = \sum_{F \in \mathbb{N}^{<\omega}} \left( \sum_{\text{supp}(\mathbf{a})=F} n_{\mathbf{a}} \right) t_F$$

where

$$\text{supp}(\mathbf{a}) = \{n \in \mathbb{N} \mid a_n \neq 0\} \text{ and } t_F = \prod_{n \in F} t_n.$$

Let  $\mathfrak{I}_1$  be the kernel of the map  $S \mapsto S'$ , *i.e.* the ideal of  $\mathcal{R}$  generated by the set  $\{S - S' \mid S \in \mathcal{R}^+\}$ . Then we have

*Proposition 1.15.* *The equinumerosity relation  $\approx$  is natural if and only if the corresponding gauge ideal  $\mathfrak{I}$  includes  $\mathfrak{I}_1$ .*

**Proof.** Let  $\mathfrak{I}$  be a gauge ideal that includes  $\mathfrak{I}_1$ . We prove that the equivalence relation defined by the relation

$$X \approx Y \iff S_X - S_Y \in \mathfrak{I}$$

is a natural equinumerosity. The principles AP, ZP, UP, PP hold because the ideal  $\mathfrak{I}$  is gauge, hence we have only to prove that if  $T$  is a natural transformation of a set  $X \in \mathbb{W}$  then  $X \approx T[X]$ , or equivalently  $S_X - S_{T[X]} \in \mathfrak{I}$ .

Let  $S'_X$  and  $S'_{T[X]}$  be the squarefree series associated to the series  $S_X$  and  $S_{T[X]}$ , respectively. We prove that  $S'_X = S'_{T[X]}$ , so the series  $S_X - S_{T[X]} = S_X - S'_X + S'_{T[X]} - S_{T[X]}$  belongs to  $\mathfrak{I}_1 \subseteq \mathfrak{I}$ . Let  $n_{\mathbf{a}} t^{\mathbf{a}}$  be a (squarefree) monomial of the series  $S'_X$ , then there exist distinct points  $P_1, \dots, P_{n_{\mathbf{a}}}$  of the set  $X$  that produce the monomial  $t^{\mathbf{a}}$ . Now consider the points  $T(P_1), \dots, T(P_{n_{\mathbf{a}}})$ , that belong to the set  $T[X]$ . The map  $T$  preserves the support of each point, hence in the series  $S'_{T[X]}$  the monomial  $t^{\mathbf{a}}$  appears with a coefficient equal to  $n_{\mathbf{a}}$  because all and only  $T(P_1), \dots, T(P_{n_{\mathbf{a}}})$  can have this support. So we conclude that  $S'_{T[X]} = S'_X$ .

Conversely, given a natural equinumerosity  $\approx$  over  $\mathbb{W}$ , let  $\mathfrak{I}$  be the ideal generated by the set  $\{S_X - S_Y \mid X \approx Y\} \cup \{t_0 - 1\}$ . The ideal  $\mathfrak{I}$  is gauge, because  $\approx$  is an equinumerosity relation, so we need only to prove that  $\mathfrak{I}$  includes the ideal  $\mathfrak{I}_1$  generated by the set  $\{S - S' \mid S \in \mathcal{R}^+\}$ .

Let  $S$  be a series of  $\mathcal{R}^+$ , and pick (not necessarily distinct) sets  $X_1, \dots, X_n \in \mathbb{W}$  such that  $S = a + S_{X_1} + \dots + S_{X_n}$ , taking care that, in each set  $X_i$ , the tuples of each support of size  $k$  be at most  $k!$ . Then each squarefree series  $S'_{X_i}$  is equal to the characteristic series of a set  $Y_i \in \mathbb{W}$ , and we can define natural transformations  $T_i: X_i \rightarrow Y_i$  in such a way that  $T_i(x)$  is a permutation of the support of  $x$ , for all  $x \in X_i$ . Hence

$$S' = a + S'_{X_1} + \dots + S'_{X_n} = a + S_{Y_1} + \dots + S_{Y_n} = a + S_{T_1[X_1]} + \dots + S_{T_n[X_n]},$$

and we conclude that the series  $S - S'$  belongs to  $\mathfrak{J}$ .  $\square$

The very same proof of Theorem 1.13 can be used to prove that there exists a biunique correspondence between *natural equinumerosities* and *gauge ideals*  $\mathfrak{J}$  including  $\mathfrak{J}_1$ , and that there exists a unique order preserving embedding  $j$  of the set  $\mathfrak{N}$  of the natural numerosities corresponding to the ideal  $\mathfrak{J}$  onto the non-negative part of  $\mathcal{R}/\mathfrak{J}$ . In the next section we shall give a complete characterization of all natural numerosities as *hypernatural numbers* from suitable ultrapowers of  $\mathbb{N}$ .

## 2. EQUINUMEROSITIES THROUGH ULTRAFILTERS

We give in this section a construction of equinumerosity relations through suitable ultrafilters.

**Definition 2.1.** Let  $\mathbb{X}$  be the set of all sequences of non-negative real numbers  $\mathbf{x} = \langle x_0, \dots, x_n, \dots \rangle$  such that the series  $\sum x_n = \|\mathbf{x}\|$  converges. For  $S \in \mathcal{R}$  let  $S(\mathbf{x})$  be the value taken by  $S$  when  $x_n$  is assigned to the variable  $t_n$ . (So e.g.  $\|\mathbf{x}\|^d = S_{\mathbb{N}^d}(\mathbf{x})$ .)

- A subset  $\mathbb{I} \subseteq \mathbb{X}$  is a *counting set* (of assignments) if for all  $k \in \mathbb{N}$  the set  $\mathbb{I}_k = \{\mathbf{i} \in \mathbb{I} \mid i_0 = i_1 = \dots = i_k = 1\} \neq \emptyset$ ;
- an ultrafilter  $\mathcal{U}$  on the counting set  $\mathbb{I}$  is *suitable for*  $\mathbb{I}$  if for all  $k \in \mathbb{N}$  the set  $\mathbb{I}_k$  is in  $\mathcal{U}$ ;
- the counting set  $\mathbb{I}$  is *suitable for* the ideal  $\mathfrak{J}$  of  $\mathcal{R}$  if for all  $S \in \mathfrak{J}$  and all  $k \in \mathbb{N}$  there exists  $\mathbf{i} \in \mathbb{I}_k$  such that  $S(\mathbf{i}) = 0$ .

Let  $\mathbb{I}$  be a counting set, and define the *counting map*

$$\Phi: \mathcal{R} \rightarrow \mathbb{R}^{\mathbb{I}} \text{ by } \Phi(S) = \langle S(\mathbf{i}) \rangle_{\mathbf{i} \in \mathbb{I}}.$$

Then  $\Phi$  is a ring homomorphism that preserves the respective partial orderings.

If  $\mathcal{U}$  is an ultrafilter suitable for  $\mathbb{I}$  and  $\pi_{\mathcal{U}}: \mathbb{R}^{\mathbb{I}} \rightarrow \mathbb{R}^{\mathbb{I}}_{\mathcal{U}}$  is the natural projection onto the corresponding ultrapower, put  $\phi_{\mathcal{U}} = \pi_{\mathcal{U}} \circ \Phi$ , so that

$$\phi_{\mathcal{U}}(S) = [\langle S(\mathbf{i}) \rangle_{\mathbf{i} \in \mathbb{I}}]_{\mathcal{U}}.$$

Then  $\phi_{\mathcal{U}}$  is a ring homomorphism whose kernel  $\mathfrak{J} = \ker \phi_{\mathcal{U}}$  is a prime ideal of  $\mathcal{R}$  that includes  $\mathfrak{J}_0$  and is disjoint from  $\mathcal{R}^+$ . Moreover the counting set  $\mathbb{I}$  turns out to be *suitable for* the ideal  $\mathfrak{J}$ .

The map  $X \mapsto \phi_{\mathcal{U}}(S_X)$  induces an equivalence relation between point sets

$$X \approx_{\mathcal{U}} Y \iff \{\mathbf{i} \in \mathbb{I} \mid S_X(\mathbf{i}) = S_Y(\mathbf{i})\} \in \mathcal{U}$$

that satisfies all conditions of an equinumerosity relation but possibly Zermelo's Principle ZP.

If the kernel  $\mathfrak{J} = \ker \phi_{\mathcal{U}}$  is a gauge ideal, then  $\approx_{\mathcal{U}}$  is an equinumerosity relation whose set of numerosities is (isomorphic to) a discrete semiring of hyperreal

numbers, namely a subsemiring of the non-negative part of the ultrapower  $\mathbb{R}_{\mathcal{U}}^{\mathbb{I}}$ . A simple rephrasing of the definition gives

**Proposition 2.2.** *The equivalence  $\approx_{\mathcal{U}}$  is an equinumerosity, or equivalently  $\mathfrak{J} = \ker \phi_{\mathcal{U}}$  is a gauge ideal of  $\mathcal{R}$ , if and only if, for all  $S \in \mathcal{R}$ ,*

$$\{\mathbf{i} \in \mathbb{I} \mid S(\mathbf{i}) > 0\} \in \mathcal{U} \iff \exists P \in \mathcal{R}^+ \text{ s.t. } \{\mathbf{i} \in \mathbb{I} \mid S(\mathbf{i}) = P(\mathbf{i})\} \in \mathcal{U}.$$

□

Conversely we can state

**Theorem 2.3.** *Let  $\approx$  be an equinumerosity, and let  $\mathfrak{J}$  be the corresponding gauge ideal of  $\mathcal{R}$ . Assume that there exists a counting set  $\mathbb{I}$  suitable for  $\mathfrak{J}$ . Then there exists an ultrafilter  $\mathcal{U}$  suitable for  $\mathbb{I}$  such that the equinumerosity  $\approx$  coincides with the equivalence  $\approx_{\mathcal{U}}$  induced by the counting map  $\phi_{\mathcal{U}}$ .*

*So the set of numerosities  $\mathfrak{N}$  of  $\approx$  is isomorphic to a discrete subsemiring of the non-negative part of the ultrapower  $\mathbb{R}_{\mathcal{U}}^{\mathbb{I}}$ .*

**Proof.** First of all remark that the family  $\mathcal{F}$  of the zero-sets  $Z(S) = \{\mathbf{i} \in \mathbb{I} \mid S(\mathbf{i}) = 0\}$  for  $S \in \mathfrak{J}$  has the finite intersection property, because  $Z(S) \cap Z(T) = Z(S^2 + T^2)$ . Let  $\mathcal{U}$  be any ultrafilter containing  $\mathcal{F}$ . We claim that if  $Z(S) \in \mathcal{U}$  then  $S \in \mathfrak{J}$ .

If  $S \notin \mathfrak{J}$  we may assume without loss of generality that there exists  $P \in \mathcal{R}^+$  and  $I \in \mathfrak{J}$  such that  $S = I + P$ . Then  $Z(S) \cap Z(I) \subseteq Z(P)$ , and  $Z(P) \cap \mathbb{I}_k = \emptyset$  for every sufficiently large  $k$ . But  $\mathbb{I}_k$  is the intersection of the zero-sets of the polynomials  $t_n - 1$  for  $n \leq k$ , so it belongs to  $\mathcal{U}$ , contradiction. Hence  $\mathfrak{J} = \ker \phi_{\mathcal{U}}$ , and the thesis follows. □

In the case of natural equinumerosities, one can consider only 0-1 assignments, because only these annihilate the series

$$\sum_{n \in \mathbb{N}} (t_n^2 - t_n)^2.$$

Namely, arrange the set of all eventually zero sequences of zeroes and ones in a sequence  $\mathbb{L} = \langle \mathbf{x}_F \mid F \in \mathbb{N}^{<\omega} \rangle$ , where  $\mathbf{x}_F(n) = 1$  if and only if  $n \in F$ . Then

**Lemma 2.4.** *The counting map  $\Phi : \mathcal{R} \rightarrow \mathbb{Z}^{\mathbb{L}}$  such that  $\Phi(S) = \langle S(\mathbf{x}_F) \mid F \in \mathbb{N}^{<\omega} \rangle$  is a surjective homomorphism of partially ordered rings, whose kernel is the ideal  $\mathfrak{J}_1$  that consists of all those series whose corresponding squarefree series is 0.*

**Proof.** The map  $\Phi$  is a homomorphism by definition, and clearly it maps  $\mathcal{R}^+$  into  $\mathbb{N}^{\mathbb{L}}$ .

Given  $S = \sum n_{\mathbf{a}} t^{\mathbf{a}} \in \mathcal{R}$  and  $F \in \mathbb{N}^{<\omega}$ , recall that  $\text{supp}(\mathbf{a}) = \{n \in \mathbb{N} \mid a_n \neq 0\}$  and  $t_F = \prod_{n \in F} t_n$ . Put  $n_F = \sum_{F=\text{supp}(\mathbf{a})} n_{\mathbf{a}}$ : then  $S(\mathbf{x}_F) = \sum_{E \subseteq F} n_E$ , and  $S' = \sum_{F \in \mathbb{N}^{<\omega}} n_F t_F$ . By the inclusion-exclusion principle one has  $n_F = \sum_{E \subseteq F} (-1)^{|F \setminus E|} S(\mathbf{x}_E)$ . Hence  $S \in \ker \Phi$  if and only if  $n_F = 0$  for all  $F \in \mathbb{N}^{<\omega}$ , or equivalently  $S' = 0$ . In particular, no non-zero squarefree series lies in the kernel of  $\Phi$ .

On the other hand, given  $g \in \mathbb{Z}^{\mathbb{L}}$  one has that any series  $S = \sum n_{\mathbf{a}} t^{\mathbf{a}}$  such that  $n_F = \sum_{E \subseteq F} (-1)^{|F \setminus E|} g(E)$  satisfies  $\Phi(S) = g$ . So  $|n_F| \leq \sum_{E \subseteq F} |g(E)|$ . Put  $d_n = \sum_{E \subseteq \{0, \dots, n\}} |g(E)|$ : then we can find in  $\mathcal{R}$  a series  $S$  of degree not exceeding  $d_n$  in each variable  $t_n$  that satisfies the condition above.

□

We are now ready to classify all natural equinumerosity relations. Call *gauge* a fine<sup>4</sup> ultrafilter  $\mathcal{U}$  over  $\mathbb{N}^{<\omega}$  if every square<sup>5</sup>  $f^2 \in \mathbb{N}^{\mathbb{L}}$  is equivalent modulo  $\mathcal{U}$  to a function  $g$  such that  $n_F = \sum_{E \subseteq F} (-1)^{|F \setminus E|} g(E) \geq 0$  for all  $F \in \mathbb{N}^{<\omega}$ . (Remark that  $\Phi(\sum n_F t_F) = g$ .)

**Theorem 2.5.** *Let  $\approx$  be a natural equinumerosity, and let  $\mathfrak{I}$  be the corresponding gauge ideal of  $\mathcal{R}$ . Then  $\mathbb{L}$  is suitable for  $\mathfrak{I}$ , and there exists a unique gauge ultrafilter  $\mathcal{U}$  over  $\mathbb{N}^{<\omega}$  suitable for  $\mathbb{L}$  such that*

$$(***) \quad X \approx Y \iff \{F \in \mathbb{N}^{<\omega} \mid S_X(\mathbf{x}_F) = S_Y(\mathbf{x}_F)\} \in \mathcal{U}.$$

Hence the counting map  $\Phi$  induces an ordered semiring isomorphism between the set of numerosities  $\mathfrak{N}$  of  $\approx$  and the ultrapower  $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}^{<\omega}}$ .

Conversely, the condition (\*\*\*) defines a natural equinumerosity on  $\mathbb{W}$  if and only if the ultrafilter  $\mathcal{U}$  is gauge.

**Proof.** The ideal  $\mathfrak{I}$  includes the ideal  $\mathfrak{I}_1$ , hence in particular all series whose corresponding squarefree is 0, i.e. the kernel of  $\Phi$ . Let  $\mathfrak{J} = \Phi[\mathfrak{I}]$  be the ideal of  $\mathbb{Z}^{\mathbb{L}}$  corresponding to  $\mathfrak{I}$ . Then  $\mathcal{R}/\mathfrak{I}$  is isomorphic to  $\mathbb{Z}^{\mathbb{L}}/\mathfrak{J}$ , and every element of  $\mathfrak{J}$  has some component equal to 0. In fact, assume that  $f \in \mathfrak{J}$  has no zeroes, and multiply  $f$  by a suitable sequence of  $\pm 1$ , so as to obtain a sequence  $g \in \mathfrak{J}$  of positive integers. Then  $g$  can be written as

$$g = 1 + h_1^2 + h_2^2 + h_3^2 + h_4^2 = \Phi(1 + S_1^2 + S_2^2 + S_3^2 + S_4^2).$$

But the series  $1 + S_1^2 + S_2^2 + S_3^2 + S_4^2$  is equivalent modulo  $\mathfrak{I}$  to an element of  $\mathcal{R}^+$ , and so its image  $g$  under  $\Phi$  cannot belong to  $\mathfrak{J}$ . So the counting set  $\mathbb{L}$  is suitable for  $\mathfrak{I}$ , and  $\mathfrak{J}$  is a prime zero-ideal that is determined by an ultrafilter  $\mathcal{U}$  over  $\mathbb{L}$ , which in turn is suitable for  $\mathbb{L}$ , because  $\mathbb{L}_k$  is the intersection of the zero-sets of the polynomials  $t_n - 1$  for  $0 \leq n \leq k$ . It follows that  $\mathcal{U}$  is a fine ultrafilter, and that the condition (\*\*\*) holds.

The ultrafilter  $\mathcal{U}$  is gauge, because, given  $f^2 = \Phi(S^2) > 0$ , there exists  $P \in \mathcal{R}^+$  that is equivalent to  $S^2$  modulo  $\mathfrak{I}$ . Put  $\Phi(P) = g$ : then clearly  $g \equiv f^2$  modulo  $\mathcal{U}$ , and satisfies the equalities  $\sum_{E \subseteq F} (-1)^{|F \setminus E|} g(E) \geq 0$  because  $P \in \mathcal{R}^+$ .

Conversely, if  $\mathcal{U}$  is a gauge ultrafilter, then (\*\*\*) defines an equinumerosity relation by Proposition 2.2, which is natural because the corresponding ideal of  $\mathcal{R}$  includes  $\mathfrak{I}_1$ .

□

In particular we have

**Corollary 2.6.** *The condition (\*\*\*) provides a biunique correspondence between natural equinumerosity relations  $\approx$  on  $\mathbb{W}$  and gauge ultrafilters  $\mathcal{U}$  on  $\mathbb{N}^{<\omega}$ , in such a way that the following diagram commutes*

$$\begin{array}{ccccc} \mathbb{W} & \xrightarrow{\Sigma} & \mathcal{R} & \xrightarrow{\Phi} & \mathbb{Z}^{\mathbb{L}} \\ \downarrow \mathfrak{n} & & \downarrow & & \downarrow \\ \mathfrak{N} & \xrightarrow{j} & \mathcal{R}/\mathfrak{I} & \xrightarrow{\cong} & \mathbb{Z}^{\mathbb{L}}/\Phi[\mathfrak{I}] \cong \mathbb{Z}_{\mathcal{U}}^{\mathbb{N}^{<\omega}} \end{array}$$

<sup>4</sup> The ultrafilter  $\mathcal{U}$  is fine if all cones  $C_n = \{F \mid n \in F\}$  belong to  $\mathcal{U}$ .

<sup>5</sup> Recall that every element of  $\mathbb{N}^{\mathbb{L}}$  is a sum of squares.

In particular all sets of natural numerosities can be taken to be sets of hypernatural numbers of the corresponding ultrapowers  $\mathbb{N}_{\mathcal{U}}^{<\omega}$ .  $\square$

*Remark 2.7.* Put  $F_k = \{0, \dots, k\}$ . In this context, the *asymptotic numerosities* defined in [5] can be characterized as the restrictions to  $\mathbb{W}_0$  of those natural equinumerosities for which  $\mathbb{I} = \{\mathbf{x}_{F_k} \mid k \in \mathbb{N}\}$  is a suitable counting set.

### 3. FINAL REMARKS AND OPEN QUESTIONS

It is interesting to remark that the natural numerosities satisfy a “Finite Approximation Principle” analogous to that considered in [2]. Let  $\mathbf{n}$  be a natural numerosity function, and for  $X \in \mathbb{W}$  and  $F \in \mathbb{N}^{<\omega}$  put  $X_F = X \cap \bigcup_{n \in \mathbb{N}} F^n$ , so that  $S_X(\mathbf{x}_F) = |X_F|$ . Then

(FAP) *If  $|X_F| \leq |Y_F|$  for all  $F \in \mathbb{N}^{<\omega}$ , then  $\mathbf{n}(X) \leq \mathbf{n}(Y)$ .*

In fact, if  $\mathcal{U}$  is the gauge ultrafilter corresponding to the numerosity  $\mathbf{n}$ , we have

$$\mathbf{n}(X) \leq \mathbf{n}(Y) \iff \{F \in \mathbb{N}^{<\omega} \mid |X_F| \leq |Y_F|\} \in \mathcal{U}.$$

This fact suggests a “Cantorian” characterization of natural equinumerosity by means of a class of particular bijections, similar to the one obtained in [5] for asymptotic equinumerosity. Call  *$\mathcal{U}$ -congruence between  $X$  and  $Y$*  a 1-to-1 map  $\tau : X \rightarrow Y$  such that  $\{F \in \mathbb{N}^{<\omega} \mid \tau[X_F] = Y_F\} \in \mathcal{U}$ . Then we have

*Remark 3.1.* Let  $\approx$  be a natural equinumerosity, and let  $\mathcal{U}$  be the corresponding gauge ultrafilter over  $\mathbb{N}^{<\omega}$ . Then, by definition,

*if there exists a  $\mathcal{U}$ -congruence between  $X$  and  $Y$ , then  $X \approx Y$ .*

The reverse implication seems difficult to prove in general. A simple inductive definition of  $\tau$  can be given whenever the ultrafilter  $\mathcal{U}$  contains a *chain*. But in this case the equinumerosity becomes asymptotic after an appropriate reordering of  $\mathbb{N}$ .

When the ultrafilter is Ramsey the situation is much simpler, namely

**Corollary 3.2.** *A fine Ramsey ultrafilter  $\mathcal{U}$  on  $\mathbb{N}^{<\omega}$  provides a natural equinumerosity relation on  $\mathbb{W}$ . The corresponding set of numerosities  $\mathbb{N}_{\mathcal{U}}^{<\omega}$  is isomorphic to the ultrapower  $\mathbb{N}_{\sigma\mathcal{U}}^{\mathbb{N}}$ , where  $\sigma : \mathbb{N}^{<\omega} \rightarrow \mathbb{N}$  is defined by  $\sigma(F) = |F|$ . This equinumerosity is characterized by the class of the  $\mathcal{U}$ -congruences, and becomes asymptotic after a suitable reordering of  $\mathbb{N}$ .*

**Proof.** Consider the coloring of  $[\mathbb{N}^{<\omega}]^2$  given by  $c(\{F, G\}) = 0$  if  $F$  is comparable with  $G$ ,  $c(\{F, G\}) = 1$  otherwise. The ultrafilter  $\mathcal{U}$  being Ramsey, it contains a homogeneous subset  $H$ , which is a chain, because  $\mathcal{U}$  is fine.

In order to prove the first assertion, we show that  $\mathcal{U}$  is gauge. Remark that,  $\mathcal{U}$  being Ramsey, any non-negative function  $f : \mathbb{N}^{<\omega} \rightarrow \mathbb{Z}$  is nondecreasing on some subchain  $H_0 \subset H_1 \subset \dots \subset H_n \subset \dots$  of  $H$  that belongs to  $\mathcal{U}$ . Define the series

$$S = f(H_0)t_{H_0} + \sum_{n \in \mathbb{N}} (f(H_{n+1}) - f(H_n))t_{H_{n+1}} \in \mathcal{R}^+ :$$

then  $\Phi(S)$  is equivalent to  $f$  modulo  $\mathcal{U}$ , and so  $\mathcal{U}$  is gauge.

Assume without loss of generality that  $H$  is complete, *i.e.* that the map  $\sigma$  restricted to  $H$  is onto  $\mathbb{N}$ . Then  $\sigma$  induces an isomorphism between the ultrapowers  $\mathbb{Z}_{\mathcal{U}}^{<\omega}$  and  $\mathbb{Z}_{\sigma\mathcal{U}}^{\mathbb{N}}$ , and the second assertion is proved.



Given equinumerous sets  $X$  and  $Y$ , let  $H_0 \subset H_1 \subset \dots \subset H_n \subset \dots$  be a subchain of  $H$  such that  $|X_{H_n}| = |Y_{H_n}|$  for all  $n \in \mathbb{N}$ . Define  $\tau : X \rightarrow Y$  by glueing together disjoint bijections  $\tau_0 : X_{H_0} \rightarrow Y_{H_0}$  and  $\tau_{n+1} : X_{H_{n+1}} \setminus X_{H_n} \rightarrow Y_{H_{n+1}} \setminus Y_{H_n}$  for all  $n \in \mathbb{N}$ . Then  $\tau$  is a  $\mathcal{U}$ -congruence between  $X$  and  $Y$ .

Finally, any complete chain  $H$  in  $\mathcal{U}$  provides a reordering of  $\mathbb{N}$  such that  $H_k$  becomes  $F_k$ : hence, with respect to this reordering, the equinumerosity is asymptotic.  $\square$

Remark that the ultrafilter  $\sigma\mathcal{U}$  on  $\mathbb{N}$  defined above is Ramsey, whenever the gauge ultrafilter  $\mathcal{U}$  contains a (complete) chain, because every non-negative function is non decreasing modulo  $\sigma\mathcal{U}$ . So natural equinumerosities exist under mild set theoretic hypotheses, namely those that provide Ramsey ultrafilters over  $\mathbb{N}$ , and the corresponding numerosities are, up to isomorphism, the *asymptotic numerosities* of [5]. The question as to whether there exist non-Ramsey gauge ultrafilters is still open, and with it the most interesting question of the existence of natural equinumerosities in ZFC alone. However we conjecture that *only P-point ultrafilters can be gauge*, and so the question would be solved in the negative. Combined with the “geometric” intuition that the size of the diagonal might be different from that of the side, this conjecture is the reason why we did not include the Natural Transformation Principle NP in the definition of equinumerosity.

On the other hand, the existence of gauge ideals of the ring  $\mathcal{R}$  seems to be weaker than that of gauge ultrafilters, and so the existence of (non-natural) equinumerosities might be provable in ZFC. However also this question is open up to now.

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